Finite tidal waves propagated without change of shape

By F. K. BALL
C.S.I.R.O. Division of Meteorological Physics, Aspendale, Victoria, Australia

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Coriolis terms are introduced into the equations governing the motion of a finite tidal wave. Various types of solution are found, all of which travel without change of shape and some which are periodic with sharp crests and broad troughs. The classical result that such waves cannot be propagated without change of shape is therefore untrue in these circumstances.

1. Introduction

The shallow water theory in its lowest approximation (i.e. tidal theory) has been applied with advantage not only to the study of long gravitational waves and tides but also to many problems of hydraulic engineering, particularly in connexion with flow in open channels. More recently applications of this type of theory have been made in meteorology (e.g. Abdullah 1949; Tepper 1950; Freeman 1951; Ball 1956, 1960), and it is certain that the usefulness of the theory in this context has not yet been adequately explored.

In tidal theory one makes the basic assumption that the vertical accelerations are sufficiently small for the vertical pressure gradient to be regarded as hydrostatic. This implies that the horizontal pressure gradient is determined solely by the inclination of the free surface. These assumptions are appropriate when the depth is small compared with some other significant length, such as for example the radius of curvature of the free surface. Now it is well known that under such conditions finite waves cannot be propagated without change of shape (e.g. see Lamb 1932, p. 278). It will be shown here that when Coriolis terms are introduced into the equations of motion and an additional force assumed so that geostrophic equilibrium can prevail, the resulting system has finite solutions which *are* propagated without change of shape; furthermore some of these solutions are periodic and represent an oscillation about geostrophic equilibrium.

In geophysical applications an additional force is acting when the fluid flows on sloping ground or when it flows as a layer beneath a lighter fluid. In the former case the force is provided by gravity and geostrophic flow is parallel to the surface contours; in the latter case the force is provided by the (horizontal) pressure gradient in the upper fluid and geostrophic flow is parallel to the isobars. Both types of flow occur, for example, on the Antarctic Ice Cap where there is frequently a layer of very cold air in contact with the ice. This cold air flows under the combined influence of gravity and the superimposed pressure gradient, gravity usually being dominant when the slope of the ice exceeds about 2×10^{-3}.

2. Basic equations

We will assume that there are two liquid layers with densities ρ_0 and ρ_1 ($\rho_0 < \rho_1$) and that the interface is at a height H above the base of the lower layer, which in turn is at a height Z above some fixed datum. The horizontal pressure gradient in the lower layer is then given by the sum of the superimposed pressure gradient in the upper layer and the gradient resulting from the slope of the interface. Thus $\nabla_1 p_1 = \nabla_1 p_0 + (\rho_1 - \rho_0) g \nabla_1 (H+Z)$ where ∇_1 denotes the horizontal gradient and the equations governing the motion of the lower layer can be written

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial X}(HU) + \frac{\partial}{\partial Y}(HV) = 0, \tag{1}$$

$$\frac{DU}{Dt} = -\frac{1}{\rho_1}\frac{\partial p_0}{\partial X} - g^* \frac{\partial}{\partial X}(H+Z) + fV, \tag{2}$$

$$\frac{DV}{Dt} = -\frac{1}{\rho_1}\frac{\partial p_0}{\partial Y} - g^* \frac{\partial}{\partial Y}(H+Z) - fU, \tag{3}$$

where g^* is the modified gravitational acceleration, $(\rho_1 - \rho_0) g/\rho_1$, and f is the Coriolis parameter. Capital letters are here used to denote some of the variables, small letters being reserved for the corresponding dimensionless quantities introduced below. We now assume further that $\nabla_1(p_0/\rho_1 + g^* Z)$ is constant (this expression represents the additional force mentioned in the introduction), and consequently the equations have a solution (geostrophic equilibrium) given by

$$U_g = -\frac{1}{f}\frac{\partial}{\partial Y}\left(\frac{p_0}{\rho_1} + g^* Z\right), \tag{4}$$

$$V_g = \frac{1}{f}\frac{\partial}{\partial X}\left(\frac{p_0}{\rho_1} + g^* Z\right), \tag{5}$$

where the equilibrium depth $H = H_g$ is constant.

We wish to investigate simple solutions of equations (2) and (3) which represent disturbances propagated without change of shape and with constant velocity c in the X-direction under the influence of a constant external force of arbitrary direction. This force is represented for convenience by the geostrophic flow U_g, V_g which it would induce. We also assume that a disturbance has negligible variation in the Y-direction, and thus $\partial H/\partial Y$, $\partial U/\partial Y$ and $\partial V/\partial Y$ are all equivalent to zero, and $\partial/\partial t$ is equivalent to $-c\,\partial/\partial X$. The equations then take the form

$$H(U-c) = Q, \tag{6}$$

$$Q\frac{dU}{dX} = -Hg^*\frac{dH}{dX} + Hf(V - V_g), \tag{7}$$

$$Q\frac{dV}{dX} = Hf(U_g - U), \tag{8}$$

where Q is a constant of integration representing physically the volume flow of the lower liquid relative to the moving system.

If we eliminate H from equations (7) and (8) by using equation (6), and make the substitutions $u = (U-c)/U_c$, $v = (V-V_g)/U_c$, $x = Xf/U_c$ and $\alpha = (U_g-c)/U_c$, where U_c is the 'critical speed' of mathematical hydraulics given by $U_c^3 = Qg^*$, then the equations have the dimensionless form

$$u\frac{du}{dx}(1-u^{-3}) = v, \tag{9}$$

$$u\frac{dv}{dx} = \alpha - u, \tag{10}$$

with the equilibrium (geostrophic) solution $v = 0, u = \alpha$. The direction of increasing X (and also x) has been so chosen that $U-c > 0$; whence, since $H > 0$, we have $Q > 0$ and $u > 0$.

The form of the solution of equations (9) and (10) depends on the value of the parameter α and the boundary condition (e.g. the condition that the solution passes through a particular point in the (u, v)-plane, the origin of x being immaterial). In the following sections all possible types of solution will be considered and the conditions under which periodic solutions can occur will be investigated in detail.

3. Solution types

The easiest way to determine the types of solution that can arise involves consideration of the 'phase portrait', i.e. solution curves in the (u, v)-plane (i.e. the 'phase plane'). To determine the form of the phase portraits we note the following facts. If we change the sign of both v and x, the form of the equations is unchanged; thus the phase portrait is symmetrical about the u-axis. The lines $u = 0$, $u = 1$, $u = \alpha$ and $v = 0$ divide the phase plane into regions within each of which du/dx and dv/dx have constant sign; furthermore on each of these lines one or other of dv/dx and du/dx is either zero or infinite. Making use of these facts it is a simple matter to sketch the phase portraits as in figure 1, the upper half plane $u > 0$ being the part of interest. There are three basically different patterns according as $\alpha > 1$, $1 > \alpha > 0$ or $0 > \alpha$. The arrows give the direction of increasing x.

The closed curves in figure 1 (a) correspond to the periodic solutions in which our main interest lies. These solutions are clearly periodic, for if one of these closed curves is followed in an anticlockwise direction then x continuously increases whereas u and v alternate regularly between their extreme values. On the other hand the closed curves in figures 1 (b) and 1 (c) do *not* correspond to *simple* periodic solutions since the direction of increasing x changes where the curves cross the line $u = 1$. The solutions corresponding to these curves form a series of closed loops in the (u, x)-plane. Furthermore it is certain that these solutions (and all others that cross the line $u = 1$ with a reversal of the direction of increasing x) are physically invalid in the neighbourhood of $u = 1$, not only because vertical accelerations are likely to be important there but also because the solution 'turns back' in a way which is physically inconsistent with our assumptions.

The lowest group of curves in figures 1(*a*) and 1(*b*) correspond to a moving trough with maximum u (minimum H) and a change in sign of v. Because of the superimposed pressure distribution the minimum of H does not necessarily coincide with minimum pressure. We do not propose to discuss this type of solution here.

The only solution which is admissible over its whole range, apart from the periodic type and the trough type mentioned above, is the singular solution, in the form of a closed loop in the phase plane, indicated by the heavier line in

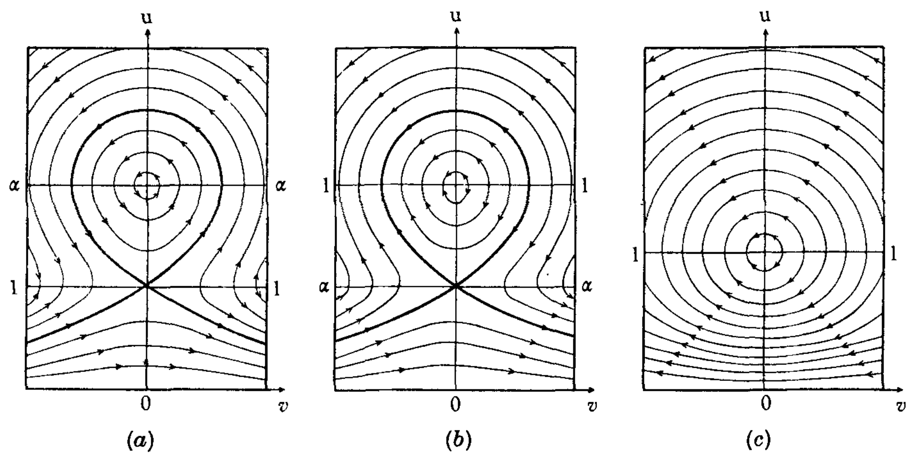

FIGURE 1. Possible phase portraits: (*a*) $\alpha > 1$; (*b*) $1 > \alpha > 0$; (*c*) $0 > \alpha$.

figure 1(*a*). This solution is unique in that it crosses the line $u = 1$ without du/dx becoming infinite. It is also important because it forms the boundary of the periodic region, and the extreme values of u and v which occur on the loop give the upper and lower bounds of u and v which are possible in a periodic solution.

4. The periodic solutions

Equations (9) and (10) can readily be integrated once to a form which gives the analytic expression for the phase portrait, viz. by multiplying equation (9) by $u-\alpha$ and equation (10) by v, adding and integrating to give

$$\frac{1}{2}\left[v^2+(u-\alpha)^2-\alpha\left(\frac{u-\alpha}{\alpha u}\right)^2\right] = E. \tag{11}$$

The constant of integration E is a measure of the energy of the motion regarded as a disturbance of the geostrophic flow $v = 0$, $u = \alpha$. The extreme values for a periodic solution are $E = 0$ corresponding to geostrophic equilibrium and $E = (\alpha-1)^3/2\alpha$ corresponding to the singular solution passing through the point $v = 0$, $u = 1$. The conditions for a simple periodic solution are therefore $\alpha > 1$, $u > 1$ and

$$\left[v^2+(u-\alpha)^2-\alpha\left(\frac{u-\alpha}{\alpha u}\right)^2\right] < \frac{(\alpha-1)^3}{\alpha}.$$

The extreme values of v occur when $u = \alpha$, and the maximum possible amplitude of v for a simple periodic solution is given by

$$|v| = \left[\frac{(\alpha-1)^3}{\alpha}\right]^{\frac{1}{2}};$$

similarly the extreme values of u occur when v is zero, whence u lies in the range

$$1 < u < (\alpha-1)+[\alpha(\alpha-1)+1]^{\frac{1}{2}}.$$

When the amplitude of the oscillation is small the solution curve in the phase plane is a small ellipse centred on the point $(0,\alpha)$. A simple analytic solution can readily be found by linearizing the equations (9) and (10). The result is

$$u-\alpha = A\cos\frac{2\pi x}{\lambda},$$
$$v = -A\left(\frac{\alpha^3-1}{\alpha^3}\right)^{\frac{1}{2}}\sin\frac{2\pi x}{\lambda}, \qquad (12)$$

where the wavelength λ is given by

$$\lambda = 2\pi\left(\frac{\alpha^3-1}{\alpha}\right)^{\frac{1}{2}}. \qquad (13)$$

An analytic solution can also be found in the other extreme when

$$E = (\alpha-1)^3/2\alpha$$

and the wave has maximum possible amplitude. We then have from equation (11)

$$v^2 = \left(\frac{u-1}{u}\right)^2[\alpha+2u(\alpha-1)-u^2],$$

whence from equation (9)

$$\frac{du}{dx} = \frac{u[\alpha+2u(\alpha-1)-u^2]^{\frac{1}{2}}}{u^2+u+1}.$$

Therefore
$$x = \int_1^u \frac{(s^2+s+1)\,ds}{s[\alpha+2s(\alpha-1)-s^2]^{\frac{1}{2}}},$$

where x is the distance from the wave crest (minimum u). Furthermore

$$\int\frac{(s^2+s+1)\,ds}{s[\alpha+2s(\alpha-1)-s^2]^{\frac{1}{2}}} = -\left\{\alpha\cos^{-1}\left(\frac{s-\alpha-1}{\beta}\right)\right.$$
$$\left.+\alpha^{-\frac{1}{2}}\ln\left|\frac{[\alpha(\alpha+2s(\alpha-1)-s^2)]^{\frac{1}{2}}+\alpha+s(\alpha-1)}{s}\right|+[\alpha+2s(\alpha-1)-s^2]^{\frac{1}{2}}\right\},$$

where $\beta^2 = \alpha(\alpha-1)+1$. Therefore the wavelength is given by

$$\lambda = 2\int_1^{\beta+\alpha-1}\frac{(s^2+s+1)\,ds}{s[\alpha+2s(\alpha-1)-s^2]^{\frac{1}{2}}}$$
$$= 2\left\{\alpha\cos^{-1}\left(\frac{2-\alpha}{\beta}\right)+\alpha^{-\frac{1}{2}}\ln\left|\frac{\{3\alpha(\alpha-1)\}^{\frac{1}{2}}+2\alpha-1}{\beta}\right|+\{3(\alpha-1)\}^{\frac{1}{2}}\right\}.$$

A graph of this function, representing the dependence of the maximum possible wavelength on α, is shown in figure 2 together with a graph of the corresponding function for waves of infinitesimal amplitude (equation (13)). When α is large we have $\lambda \simeq 2\pi\alpha$ for both infinitesimal and maximum amplitude. It appears therefore that there is little change in wavelength with variation in amplitude.

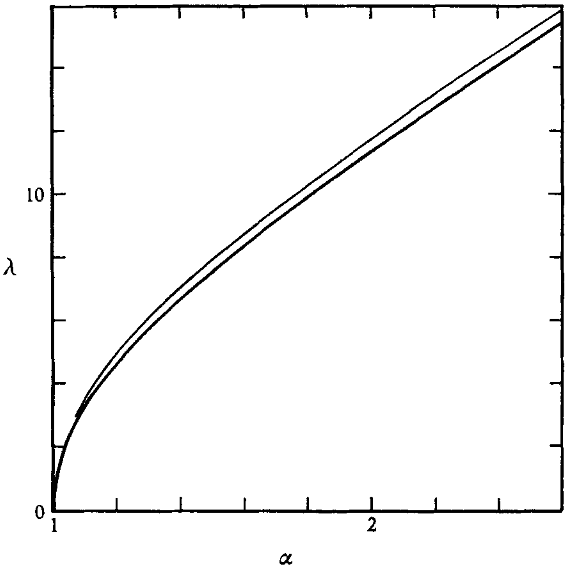

FIGURE 2. Wavelength as a function of α. Upper curve: infinitesimal wave. Lower curve: wave of maximum amplitude.

To express the actual wavelength L for the infinitesimal wave in terms of the physical variables, we note that

$$L = \frac{\lambda U_c}{f} = \frac{2\pi}{f}\left[(U_g-c)^2 - \frac{Qg^*}{U_g-c}\right]^{\frac{1}{2}}.$$

The equilibrium depth is given by $H_g = Q/(U_g-c)$, whence

$$L = \frac{2\pi}{f}[(U_g-c)^2 - H_g g^*]^{\frac{1}{2}}. \tag{14}$$

When the wavelength is small this reduces to the usual expression for the velocity of tidal waves.

More detailed investigation reveals that the waves have the following properties. The wave of infinitesimal amplitude is harmonic in form. As the amplitude increases the wavelength decreases slightly, the troughs become broad and shallow and the crests narrow and sharp. The cross wave component changes rapidly from one extreme to another on the passage of a sharp crest. At maximum amplitude the wave crests become pointed, the points becoming so sharp for large values of α that the waves are almost cusped. Some of these properties are illustrated in figure 3. For completeness an 'inadmissible' periodic solution, corresponding to a closed curve in figure 1 (*b*), is illustrated in figure 4. As mentioned in § 3, this and other solutions are physically invalid in the neighbourhood of $u = 1$ (this does not, however, imply that the other parts of these solutions are necessarily invalid).

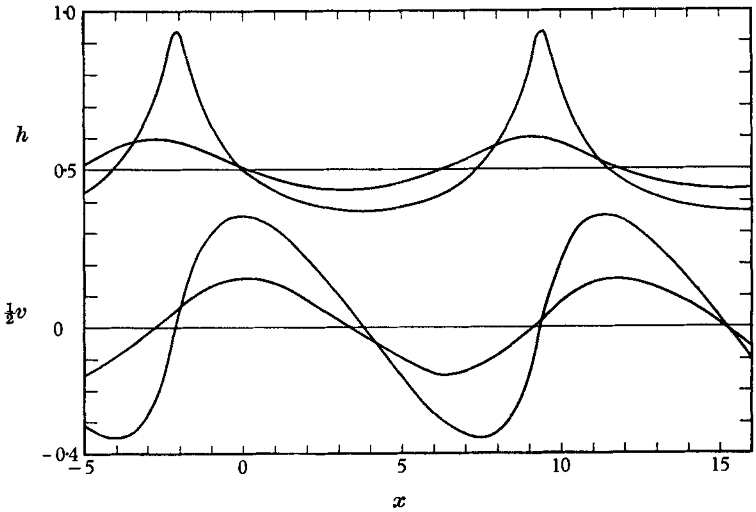

FIGURE 3. Typical wave forms; $\alpha = 2$. Upper curves: profiles of h ($= u^{-1}$) for large and small amplitude; lower curves: corresponding profiles of $\frac{1}{2}v$.

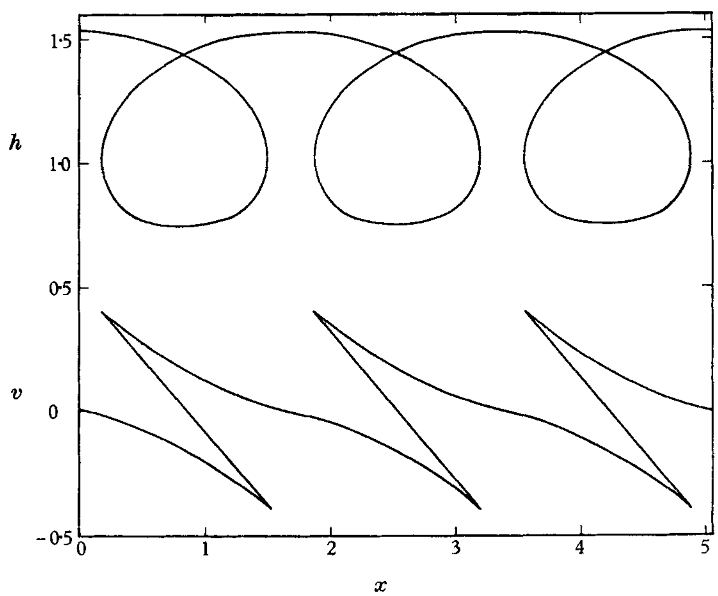

FIGURE 4. An 'inadmissible' periodic solution; $\alpha = 0{\cdot}5$. Upper curve: profile of h ($= u^{-1}$); lower curve: profile of v.

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